Math 312

MELROSE **FINAL- Sample**

1. For the following system of linear-equations, determine the values of a and b that give (i) no solution, (ii) a unique solution, (iii) many solutions.
Find the solutions for case (iii).

$$\begin{array}{rcrcrcl} x & + & 2y & - & z & = & 1 \\ -x & + & y & + & 7z & = & b \\ 2x & + & 5y & + & az & = & 3 \end{array}$$

$\boxed{(i)\ a=0,\ b\neq 2\ (ii)\ a\neq 0,\ b \text{ arbitrary},\ (iii)\ a=0,\ b=2.\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}}$

2. Find all the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

$\boxed{\lambda = 0, 4, 4.\ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$

3. By making the change of variables $u = xy$, $v = x^2$, solve the partial differential equation

$$xyf_{xy} - y^2 f_{yy} - yf_y = x^2$$

$$\boxed{f(x,y) = \tfrac{x^2}{2}\ln|xy| + g(xy) + h(x^2)}$$

4. (a) Use the method of Lagrange multipliers to find the extrema of $g(x,y) = x^2 - y^2$ subject to $4x^2 + 2y^2 = 1$.
$\boxed{g\left(0, \tfrac{\pm 1}{\sqrt{2}}\right) = -\tfrac{1}{2} \text{ min}}$, $\boxed{g\left(\tfrac{\pm 1}{2}, 0\right) = \tfrac{1}{4} \text{ max}}$

(b) Find and identify the critical points of $f(x,y) = 2x^3 + (x-y)^2 - 6y$
$\boxed{(1,4)\text{min.}, (-1,2)\text{saddle pt are } C.P.'s}$

5. (a) Does the equation $(x^2+y^2+z^2)^{1/2} = \cos z$ define $y = y(x,z)$ in the neighborhood of $(0,1,0)$?
If so, evaluate $\dfrac{\partial y}{\partial x}(0,0)$ $\boxed{\text{Yes, } \dfrac{\partial y}{\partial x} = 0 \ @(0,1,0)}$

(b) Does the equation $(x^2+y^2+z^2)^{1/2} = \cos z$ define $z = z(x,y)$ in the neighborhood of $(0,1,0)$?
If so, evaluate $\dfrac{\partial z}{\partial x}(0,1)$ $\boxed{\text{No, } z \neq z(x,y) \text{ in the neighborhood of } (0,1,0)}$

6. By making the change of variables $u = y$, $v = y - 2x$, or otherwise, evaluate

$$\int_0^\infty \int_{2x}^\infty (y-2x)^2 e^{-y} dy dx \boxed{= 1}$$

7. Use spherical coordinates to evaluate $\iiint_V z dV$ where V is the volume enclosed by the cone $\phi = \frac{\pi}{3}$ and the sphere $\rho = 2$. $\boxed{= 3\pi}$

8. Calculate the surface integral $\iint_S \mathbf{v} \cdot \mathbf{n} dS$ where $\mathbf{v} = (2x, 2y, 1)$ and S is the closed surface bounded by $z = 1 - x^2 - y^2$ and $z = 0$. Directly: $\iint_S \mathbf{v} \cdot \mathbf{n} dS$ $\boxed{3\pi - \pi = 2\pi}$

   By the divergence theorem.

9. Evaluate by any method $\oint_C y^2 dx + xy dy$ where C is the straight line segment from $(-1, 0)$ to $(1, 0)$ followed by the semi-ellipse $x^2 + 4y^2 = 1$ $y \geq 0$. $\boxed{-\frac{1}{6}}$

10. If C is the curve of intersection of the paraboloid $z = 2 - x^2 - y^2$ and the plane $z = 1$ and $\mathbf{F} = (-y, x, xyz)$, calculate $\int_C \mathbf{F} \cdot \mathbf{dr}$

    Directly:
    By Stoke's theorem: $\boxed{2\pi.}$